Galactic metric, dark radiation, dark pressure and gravitational lensing in brane world models

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ABSTRACT

In the braneworld scenario, the four dimensional effective Einstein equation has extra terms which arise from the embedding of the 3-brane in the bulk. These non-local effects, generated by the free gravitational field of the bulk, may provide an explanation for the dynamics of the neutral hydrogen clouds at large distances from the galactic center, which is usually explained by postulating the existence of the dark matter. Starting from the observational fact of the constancy of the galactic rotation curves in spiral galaxies, we obtain the exact galactic metric in the flat rotation curves region in the brane world scenario. The basic physical parameters describing the non-local effects (dark radiation and dark pressure) are obtained, and their behavior is considered in detail. Due to the presence of the bulk effects, the flat rotation curves could extend several hundred kpc. The limiting radius for which bulk effects are important is estimated and compared with the numerical values of the truncation parameter of the dark matter halos, obtained from weak lensing observations. There is a relatively good agreement between the predictions of the model and observations. The deflection of photons passing through the region where the rotation curves are flat is also considered and the bending angle of light is computed. The bending angle predicted by the brane world models is much larger than that predicted by standard general relativistic and dark matter models. The angular radii of the Einstein rings are obtained in the small angles approximation. The predictions of the brane world model for the tangential shear are compared with the observational data obtained in the weak lensing of galaxies in the Red-Sequence Cluster Survey. Therefore the study of the light deflection by galaxies and the gravitational lensing could discriminate between the different dynamical laws proposed to model the motion of particles at the galactic level and the standard dark matter models.

Subject headings: galactic rotation curves: gravitational lensing: brane world models

1. Introduction

The rotation curves for galaxies or galaxy clusters should, according to Newton's gravitation theory, show a Keplerian decrease with distance r of the orbital rotational speed v_{tg} at the rim of the luminous matter, $v_{tg}^2 \propto M(r)/r$, where $M(r)$ is the dynamical mass. However, one observes instead rather flat rotation curves (Rubin et al. 1980; Binney and Tremaine 1987; Persic et al. 1996; Boriello and Salucci 2001). Observations show that the rotational velocities increase near the center of the galaxy and then remain nearly constant at a value of $v_{tg\infty} \sim 200-300$ km/s. This leads to a general mass profile $M(r) \approx rv_{tg\infty}^2/G$ (Binney and Tremaine 1987). Consequently, the mass within a distance r from the center of the galaxy increases linearly with r, even at large distances, where very little luminous matter can be detected.

This behavior of the galactic rotation curves is explained by postulating the existence of some dark (invisible) matter, distributed in a spherical halo around the galaxies. The dark matter is assumed to be a cold, pressureless medium. There are many possible candidates for dark matter, the most popular ones being the weekly interacting massive particles (WIMP) (for a recent review of the particle physics aspects of dark matter see Overduin and Wesson (2004)). Their interaction cross section with normal baryonic matter, while extremely small, are expected to be non-zero and we may expect to detect them directly. It has also been suggested that the dark matter in the Universe might be composed of superheavy particles, with mass $\geq 10^{10}$ GeV. But observational results show the dark matter can be composed of superheavy particles only if these interact weakly with normal matter or if their mass is above 10^{15} GeV (Albuquerque and Baudis 2003). Scalar fields or other long range coherent fields coupled to gravity have also intensively been used to model galactic dark matter (Nucamendi et al. 2000; Matos and Guzman 2001; Mielke and Schunk 2002; Cabral-Rosetti et al. 2002; Lidsey et al. 2002; Mbelek 2004; Matos et al. 2004; Fuchs and Mielke 2004; Lee and Lee 2004; Mielke and Peralta 2004; Hernandez et al. 2004).

However, despite more than 20 years of intense experimental and observational effort, up to now no *non-gravitational* evidence for dark matter has ever been found: no direct evidence of it and no annihilation radiation from it. Moreover, accelerator and reactor experiments do not support the physics (beyond the standard model) on which the dark matter hypothesis is based.

Therefore, it seems that the possibility that Einstein's (and the Newtonian) gravity breaks down at the scale of galaxies cannot be excluded *a priori*. Several theoretical models, based on a modification of Newton's law or of general relativity, have been proposed to explain the behavior of the galactic rotation curves. A modified gravitational potential of the form $\phi = -GM\left[1+\alpha\exp\left(-r/r_0\right)\right]/\left(1+\alpha\right)r$, with $\alpha = -0.9$ and $r_0 \approx 30$ kpc can

explain flat rotational curves for most of the galaxies (Sanders 1984, 1986).

In an other model, called MOND, and proposed by Milgrom (Milgrom 1983; Bekenstein and Milgrom 1984; Milgrom 2002, 2003), the Poisson equation for the gravitational potential $\nabla^2\phi = 4\pi G\rho$ is replaced by an equation of the form $\nabla\left[\mu\left(x\right)\left(\left|\nabla\phi\right|/a_0\right)\right] = 4\pi G\rho$, where a_0 is a fixed constant and $\mu\left(x\right)$ a function satisfying the conditions $\mu\left(x\right) = x$ for $x << 1$ and $\mu\left(x\right) = 1$ for $x >> 1$. The force law, giving the acceleration a of a test particle becomes $a = a_N$ for $a_N >> a_0$ and $a = \sqrt{a_N a_0}$ for $a_N << a_0$, where a_N is the usual Newtonian acceleration. The rotation curves of the galaxies are predicted to be flat, and they can be calculated once the distribution of the baryonic matter is known. A relativistic MOND inspired theory was developed by Bekenstein (Bekenstein 2004, 2005). In this theory gravitation is mediated by a metric, a scalar field and a 4-vector field, all three dynamical.

Alternative theoretical models to explain the galactic rotation curves have been proposed by Mannheim (1993), Mannheim (1997), Moffat and Sokolov (1996), Moffat (2004) and Roberts (2004).

The detection by Brainerd et al. (1996) of weak, tangential distortion of the images of cosmologically distant, faint galaxies due to gravitational lensing by foreground galaxies has opened a new possibility of testing the alternative theories of gravitation and the dark matter hypothesis. Background galaxies are observed to be tangentially aligned around foreground galaxies because of the latter population's gravitational lensing effect. The angular dependence of the shear signal is consistent with the hypothesis that galaxies are dominated by approximately isothermal haloes, but can also be explained by assuming some phenomenological modifications of the effective (Newtonian) gravitational force at large distances (Mortlock and Turner 2001).

Galaxy-galaxy lensing could provide very powerful constraints for the alternative gravitational theories (for a general review of week lensing and its applications see Hoekstra et al. (2002)). The measurements rely on averaging over many background sources and the signal is only appreciable at large angular separations from the foreground deflectors. Thus, in the absence of dark matter, the foreground galaxies can be regarded as simple lenses, and the observational data can be used to constrain the deflection law directly. A weak-lensing mass reconstruction of the interacting cluster 1E 0657-558 has been presented in Clowe et al. (2004).

Weak lensing has been mainly used to discuss and constrain MOND (Gavazzi 2002; Hoekstra et al. 2002; Clowe et al. 2004). Gavazzi (2002) used the cluster of galaxies MS2137-23, which presents the most constrained lensing configuration of gravitational images ever detected in a distant cluster of galaxies to constrain MOND type models. According to this

analysis, the MOND model is not compatible with the observations. The need for much more baryons in the MOND model than for the dark matter paradigm implies significant dynamical differences between these models, which can be explored at very large radial distance. Observational evidence against this model was also reported in Clowe et al. (2004). For the interacting cluster 1E 0657-558 the observed offsets of the lensing mass peaks from the X-ray gas directly demonstrate the presence, and dominance, of dark matter in this cluster. However, this proof of dark matter existence holds true even under the assumption of MOND. Based on the observed gravitational shear-optical light ratios and the mass peak-X-ray gas offsets, the "dark matter" component in a MOND regime, that is, the dynamical MOND mass M_m, related to the dynamical Newtonian mass M_d by the relation $M_m(r) = M_d/\sqrt{1+(a_0/a)^2}$, would have a total mass that is at least equal to the baryonic mass of the galactic system. The observed shear and derived mass of the subclump are significantly higher than could be produced by an isothermal sphere with a 212 km/s.

Weak gravitational lensing is a very promising method for the study of the galactic dark matter halos. Results of the studies of weak lensing by galaxies have been published recently by Hoekstra et al. (2003, 2004). Using weak lensing the flattening of the galactic dark matter halos was observed. These results suggest that the dark matter halos are rounder than the light distribution, with a halo ellipticity of the order of $e_{\rm halo} \approx 0.33$. The average mass profile around galaxies was also studied in the framework of two halo models, the truncated isothermal sphere model and the Navarro-Frenk-White model. From the point of view of the alternative theories of gravitation, the main implications of these results is that spherical halos are excluded with 99% confidence, since most of the alternative theories predict an almost isotropic weak lensing signal, which is not observed.

In a series of recent papers (Harko and Mak 2004; Mak and Harko 2004; Harko and Mak 2005), several classes of conformally symmetric solutions of the static gravitational field equations in the brane world scenario (Randall and Sundrum 1999a,b), in which our Universe is identified to a domain wall in a 5-dimensional anti-de Sitter space-time, have been obtained (for a review of brane world models see Maartens (2004)). The static vacuum gravitational field equations on the brane depend on the generally unknown Weyl stresses in the bulk (a higher dimensional space-time, in which our universe is embedded), which can be expressed in terms of two functions, called the dark radiation U and the dark pressure P (the projections of the Weyl curvature of the bulk, generating non-local brane stresses) (Mak and Harko 2004; Maartens 2004; Harko and Mak 2005). Generally, the vacuum field equations on the brane can be reduced to a system of two ordinary differential equations, which describe all the geometric properties of the vacuum as functions of the dark pressure and dark radiation terms (Harko and Mak 2004). In order to close the system a functional relation between U and P is necessary.

In Harko and Mak (2004), Mak and Harko (2004) and Harko and Mak (2005) the solutions of the gravitational field equations on the brane have been obtained by using some methods from Lie group theory. As a group of admissible transformations the one-parameter group of conformal motions has been considered. The main advantage of imposing geometric self-similarity via a group of conformal motions is that this condition also uniquely fixes the mathematical form of the dark radiation and dark pressure terms, respectively, which describe the non-local effects induced by the gravitational field of the bulk. Thus there is no need to impose an arbitrary relation between the dark radiation and the dark pressure.

As a possible physical application of the solutions of the spherically symmetric gravitational field equations in the vacuum on the brane the behavior of the angular velocity v_{tg} of the test particles in stable circular orbits has been considered (Mak and Harko 2004; Harko and Mak 2005). The conformal factor ψ, together with two constants of integration, uniquely determines the rotational velocity of the particle. In the limit of large radial distances and for a particular set of values of the integration constants the angular velocity tends to a constant value. This behavior is typical for massive particles (hydrogen clouds) outside galaxies (Binney and Tremaine 1987; Persic et al. 1996; Boriello and Salucci 2001), and is usually explained by postulating the existence of the dark matter.

Thus, the rotational galactic curves can be naturally explained in brane world models without introducing any additional hypothesis. The galaxy is embedded in a modified, spherically symmetric geometry, generated by the non-zero contribution of the Weyl tensor from the bulk. The extra-terms, which can be described as the dark radiation term U and the dark pressure term P, act as a "dark matter" distribution outside the galaxy. The existence of the dark radiation term generates an equivalent mass term M_U, which is linearly increasing with the distance, and proportional to the baryonic mass of the galaxy, $M_U(r) \approx M_B(r/r_0)$. The particles moving in this geometry feel the gravitational effects of U, which can be expressed in terms of an equivalent mass. Moreover, the limiting value $v_{tg\infty}$ of the angular velocity can be obtained as function of the total baryonic mass M_B and radius r_0 of the galaxy as $v_{tg\infty} \approx (2/\sqrt{3})\sqrt{GM_B/r_0} + (1/12\sqrt{3})\,(GM_B/r_0)^{3/2}$ (Mak and Harko 2004).

For a galaxy with baryonic mass of the order $10^9 M_\odot$ and radius of the order of $r_0 \approx 70$ kpc, we have $v_{tg\infty} \approx 287$ km/s, which is of the same order of magnitude as the observed value of the angular velocity of the galactic rotation curves. In the framework of this model all the relevant physical parameters (metric tensor components, dark radiation and dark pressure terms) can be obtained as functions of the tangential velocity, and hence they can be determined observationally.

However, imposing a group of conformal motions on the brane implies a major restriction on the geometrical structure of the space-time. Therefore it would be important to analyze

the behavior of the galactic rotation curves in the brane world model without particular assumptions on the geometry of the brane. In the framework of a Newtonian approximation of the brane metric the possibility that the galactic rotation curves can be explained by the presence of the dark radiation and dark pressure was considered in Pal et al. (2005).

It is the purpose of this paper to consider the geometric properties of the space-time at the galactic level on the brane, and to derive the expressions of the physical quantities (dark radiation and dark pressure), which determine the dynamic of the particle in circular orbit. Under the assumption of spherical symmetry the basic equations describing the static gravitational field on the brane depend on two unknown parameters, the dark radiation and the dark pressure. As a starting point in our study we adopt the well-established observation of the constancy of the galactic rotation curves far-away from the galactic center. This property allows the immediate determination of the exact galactic metric on the brane and, consequently, of the mathematical form of the dark radiation and dark pressure terms. The behavior of other physical parameters (effective potential, angular momentum and energy) is also considered.

As a direct observational application of the obtained results we consider the bending of light and the lensing by galaxies in the flat rotation curves region. It has already been pointed out that braneworld black holes could have significantly different gravitational lensing observational signatures as compared to the Schwarzschild black holes (Whisker 2005). A general expression for the bending angle in the flat rotation curves region is derived and the deflection of light is studied numerically. The deflection angle depends on the tangential velocity of the particles in stable circular orbits, the baryonic mass and the radius of the galaxy. An analytic expression for the deflection angle in the first order of approximation is also obtained. The size of the radius for which the effects of the extra-dimensions are important is also derived as a function of the tangential velocity and of the cosmological parameters. The explicit expressions of observationally important parameters, like the tangential shear are presented. Hence the theoretical predictions of the deflection of light in the brane world models can be compared with the observations.

The predictions of the brane world model are compared with the observational data in two cases. First, we compare the observational values of the truncation parameter (the size of the dark matter halos), obtained in the framework of the truncated isothermal sphere model by Hoekstra et al. (2004) with the size of the region for which bulk effect are important on the brane. There is a relatively good agreement between these two quantities, the difference between prediction and observation being in the range of $15-20\%$. Secondly, we compare the tangential shear in the present model with the observational values of Hoekstra et al. (2004). Consistency between brane world models and observations is obtained if the mass-radius

ratio of the galactic baryonic matter is of the order of $10^{-4.5} - 10^{-4}$. Therefore the bending of light could provide a powerful method to distinguish between models which assume that dark matter is a form of unknown matter or is a result of a change in the dynamical laws that govern the motion of particles.

The introduction in the past few years of new observational techniques moved cosmology into the era of precision science. From the study of the cosmic microwave background (CMB), large scale structure of galaxies (LSS) and distant type Ia supernovae, a new paradigm of cosmology has been established. In this new standard model, the geometry of the Universe is flat so that $\Omega_{\rm total} = 1$, and the total density is made up of matter, (comprised of baryons ($\Omega_b = 0.005$ and cold dark matter ($\Omega_{CDM} = 0.25$) and dark energy with $\Omega_\Lambda = 0.7$ (Tegmark et al. 2004). In particular, current cosmological data provides a very precise bound on the physical dark matter density, $\Omega_{\rm dm} h^2 = 0.115 \pm 0.012$ (Tegmark et al. 2004). This bound provides a very strong input on any particle physics or alternative gravitation model for dark matter. On the other hand, dark matter is assumed to play a very important role in galaxy and large scale structure formation. Since in the model developed in the present paper the basic properties of the galactic rotation curves are explained without resorting to dark matter, it seems that this would imply that all the dark matter in the Universe could be related to extra-dimensional or purely non-standard gravitational effects, as it has been already proposed (Cembranos et al. 2003; Maroto 2004). In these models the branons, hypothetical particles attached to the scalar fluctuations of the brane, are decoupled from standard model matter, thus playing the role of the dark matter. Therefore, a particle component of the cosmological dark matter cannot be excluded *a priori* even in the framework of brane world models. Moreover, non-baryonic particles from the standard model or some of its extensions could also contribute substantially to the cosmological dynamics.

The present paper is organized as follows. The gravitational field equations for a static, spherically symmetric vacuum brane are written down in Section II. The definition and main properties of the observationally important parameters for the study of the galactic rotation curves are presented in Section III. In Section IV we derive the galactic metric and the basic physical parameters (dark radiation, dark pressure, effective potential etc.) on the brane. The deflection of light in the flat rotation curves region is considered in Section V. In Section VI we discuss and conclude our results.

2. The field equations for a static, spherically symmetric vacuum brane

We start by considering a 5-dimensional space-time (the bulk), with a single 4-dimensional brane, on which gravity is confined. The 4−D brane world $\left({}^{(4)}M, g_{\mu\nu}\right)$ is located at a hy-

persurface $\left(B\left(X^{A}\right)=0\right)$ in the 5−D bulk space-time $\left({}^{(5)}M, g_{AB}\right)$, of which coordinates are described by $X^{A}, A=0,1,...,4$. The action of the system is given by (Shiromizu et al. 2000)

$$S = S_{bulk} + S_{brane}, \tag{1}$$

where

$$S_{bulk} = \int_{{}^{(5)}M} \sqrt{-{}^{(5)}g}\left[\frac{1}{2k_5^2}{}^{(5)}R + {}^{(5)}L_m + \Lambda_5\right] d^5X, \tag{2}$$

and

$$S_{brane} = \int_{{}^{(4)}M} \sqrt{-{}^{(5)}g}\left[\frac{1}{k_5^2}K^{\pm} + L_{brane}\left(g_{\alpha\beta}, \psi\right) + \lambda_b\right] d^4x, \tag{3}$$

where $k_5^2 = 8\pi G_5$ is the 5-D gravitational constant, ${}^{(5)}R$ and ${}^{(5)}L_m$ are the 5−D scalar curvature and matter Lagrangian in the bulk, respectively. $x^{\mu}, \mu = 0,1,2,3$ are the induced 4−D coordinates of the brane, $K^{\pm}$ is the trace of the extrinsic curvature on either side of the brane and $L_{brane}\left(g_{\alpha\beta}, \psi\right)$ is the 4−D Lagrangian, which is given by a generic functional of the brane metric $g_{\alpha\beta}$ and of the matter fields ψ. In the following capital Latin indices run in the range $0,...,4$, while Greek indices take the values $0,...,3$. Λ_5 and Λ are the negative vacuum energy densities in the bulk and in the brane, respectively.

The Einstein field equations in the bulk are given by (Shiromizu et al. 2000; Sasaki et al. 2000; Maeda et al. 2003)

$${}^{(5)}G_{IJ} = k_5^{2\,(5)}T_{IJ}, \qquad {}^{(5)}T_{IJ} = -\Lambda_5{}^{(5)}g_{IJ} + \delta(B)\left[-\lambda_b{}^{(5)}g_{IJ} + T_{IJ}\right], \tag{4}$$

where

$${}^{(5)}T_{IJ} \equiv -2\frac{\delta^{(5)}L_m}{\delta^{(5)}g^{IJ}} + {}^{(5)}g_{IJ}{}^{(5)}L_m, \tag{5}$$

is the energy-momentum tensor of bulk matter fields, while $T_{\mu\nu}$ is the energy-momentum tensor localized on the brane and which is defined by

$$T_{\mu\nu} \equiv -2\frac{\delta L_{brane}}{\delta g^{\mu\nu}} + g_{\mu\nu}L_{brane}. \tag{6}$$

The delta function $\delta(B)$ denotes the localization of brane contribution. In the 5-D space-time a brane is a fixed point of the Z_2 symmetry. The basic equations on the brane are obtained by projection of the variables onto the brane world. The induced 4−D metric is $g_{IJ} = {}^{(5)}g_{IJ} - n_I n_J$, where n_I is the space-like unit vector field normal to the brane hypersurface ${}^{(4)}M$. In the following we assume that all the matter fields, except gravitation, are confined to the brane. This implies that ${}^{(5)}L_m = 0$.

Assuming a metric of the form $ds^2 = (n_I n_J + g_{IJ})dx^I dx^J$, with $n_I dx^I = d\chi$ the unit normal to the χ =constant hypersurfaces and g_{IJ} the induced metric on χ =constant hypersurfaces, the effective four-dimensional gravitational equations on the brane (the Gauss equation), take the form:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, \tag{7}$$

where $S_{\mu\nu}$ is the local quadratic energy-momentum correction

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_\mu{}^\alpha T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu}\left(3T^{\alpha\beta}T_{\alpha\beta} - T^2\right), \tag{8}$$

and $E_{\mu\nu}$ is the non-local effect from the free bulk gravitational field, the transmitted projection of the bulk Weyl tensor C_{IAJB}, $E_{IJ} = C_{IAJB}n^A n^B$, with the property $E_{IJ} \to E_{\mu\nu}\delta_I^\mu \delta_J^\nu$ as $\chi \to 0$ (Shiromizu et al. 2000; Sasaki et al. 2000; Maeda et al. 2003). We have also denoted $k_4^2 = 8\pi G$, with G the usual four-dimensional gravitational constant.

The four-dimensional cosmological constant, Λ, and the four-dimensional gravitational coupling constant, k_4, are given by $\Lambda = k_5^2\left(\Lambda_5 + k_5^2 \lambda_b^2/6\right)/2$ and $k_4^2 = k_5^4 \lambda_b/6$, respectively. In the limit $\lambda_b^{-1} \to 0$ we recover standard general relativity.

The Einstein equation in the bulk and the Codazzi equation also imply the conservation of the energy-momentum tensor of the matter on the brane, $D_\nu T_\mu{}^\nu = 0$, where D_ν denotes the brane covariant derivative. Moreover, from the contracted Bianchi identities on the brane it follows that the projected Weyl tensor should obey the constraint $D_\nu E_\mu{}^\nu = k_5^4 D_\nu S_\mu{}^\nu$.

The symmetry properties of $E_{\mu\nu}$ imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field u^μ as (Maartens 2004)

$$E_{\mu\nu} = -\tilde{k}^4\left[U\left(u_\mu u_\nu + \frac{1}{3}h_{\mu\nu}\right) + P_{\mu\nu} + 2Q_{(\mu}u_{\nu)}\right], \tag{9}$$

where $\tilde{k} = k_5/k_4$, $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects orthogonal to u^μ, the "dark radiation" term $U = -\tilde{k}^4 E_{\mu\nu}u^\mu u^\nu$ is a scalar, $Q_\mu = k_4^4 h_\mu^\alpha E_{\alpha\beta}$ a spatial vector and $P_{\mu\nu} = -\tilde{k}^4\left[h_{(\mu}{}^\alpha h_{\nu)}{}^\beta - \frac{1}{3}h_{\mu\nu}h^{\alpha\beta}\right]E_{\alpha\beta}$ a spatial, symmetric and trace-free tensor.

In the case of the vacuum state we have $\rho = p = 0$, $T_{\mu\nu} \equiv 0$ and consequently $S_{\mu\nu} = 0$. Therefore the field equations describing a static brane take the form

$$R_{\mu\nu} = \Lambda g_{\mu\nu} - E_{\mu\nu}, \tag{10}$$

with the trace R of the Ricci tensor $R_{\mu\nu}$ satisfying the condition $R = R_\mu^\mu = 4\Lambda$. $E_{\mu\nu}$ is a traceless tensor, $E_\mu^\mu = 0$.

In the vacuum case $E_{\mu\nu}$ satisfies the constraint $D_\nu E_\mu{}^\nu = 0$. In an inertial frame at any point on the brane we have $u^\mu = \delta_0^\mu$ and $h_{\mu\nu} =$diag$(0,1,1,1)$. In a static vacuum $Q_\mu = 0$ and the constraint for $E_{\mu\nu}$ takes the form (Germani and Maartens 2001)

$$\frac{1}{3}D_\mu U + \frac{4}{3}UA_\mu + D^\nu P_{\mu\nu} + A^\nu P_{\mu\nu} = 0, \tag{11}$$

where D_μ is the projection (orthogonal to u^μ) of the covariant derivative and $A_\mu = u^\nu D_\nu u_\mu$ is the 4-acceleration.

In the static spherically symmetric case we may chose $A_\mu = A(r)r_\mu$ and $P_{\mu\nu} = P(r)\left(r_\mu r_\nu - \frac{1}{3}h_{\mu\nu}\right)$, where $A(r)$ and $P(r)$ (the "dark pressure") are some scalar functions of the radial distance r, and r_μ is a unit radial vector (Dadhich et al. 2000).

We choose the static spherically symmetric metric on the brane in the standard form

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{12}$$

Then the gravitational field equations and the effective energy-momentum tensor conservation equation in the vacuum take the form (Harko and Mak 2004; Mak and Harko 2004; Harko and Mak 2005)

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = \frac{48\pi G}{k_4^4\lambda_b}U + \Lambda, \tag{13}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{16\pi G}{k_4^4\lambda_b}(U + 2P) - \Lambda, \tag{14}$$

$$e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) = \frac{32\pi G}{k_4^4\lambda_b}(U - P) - 2\Lambda, \tag{15}$$

$$\nu' = -\frac{U' + 2P'}{2U + P} - \frac{6P}{r(2U + P)}, \tag{16}$$

where we denoted $' = d/dr$.

Eq. (13) can immediately be integrated to give

$$e^{-\lambda} = 1 - \frac{C_1}{r} - \frac{2GM_U(r)}{r} - \frac{\Lambda}{3}r^2, \tag{17}$$

where C_1 is an arbitrary constant of integration, and we denoted

$$M_U(r) = \frac{24\pi}{k_4^4\lambda_b}\int_0^r r^2 U(r)\,dr. \tag{18}$$

The function $M_U(r)$ is the gravitational mass corresponding to the dark radiation term (the dark mass). For $U = 0$ and $\Lambda = 0$ the metric coefficient given by Eq. (17) must tend to the standard general relativistic Schwarzschild metric coefficient, which gives $C_1 = 2GM$, where $M =$ constant is the mass of the gravitating body.

3. Stable circular orbits and frequency shifts in static space-times on the brane

The galactic rotation curves provide the most direct method of analyzing the gravitational field inside a spiral galaxy. The rotation curves have been determined for a great number of spiral galaxies. They are obtained by measuring the frequency shifts z of the light emitted from stars and from the 21-cm radiation emission from the neutral gas clouds. Usually the astronomers report the resulting z in terms of a velocity field v_{tg} (Binney and Tremaine 1987; Persic et al. 1996; Boriello and Salucci 2001).

The starting point in the analysis of the motion of the stars on the brane is to assume, as usual, that stars behave like test particles, which follow geodesics of a static and spherically symmetric space-time. Next, we consider two observers O_E and O_∞, with four-velocities u_E^μ and u_∞^μ, respectively. Observer O_E corresponds to the light emitter (i. e., to the stars placed at a point P_E of the space-time on the brane), and O_∞ represent the detector at point P_∞ located far from the emitter and that can be idealized to correspond to "spatial infinity".

Without loss of generality, we can assume that the stars move in the galactic plane $\theta = \pi/2$, so that $u_E^\mu = \left(\dot{t}, \dot{r}, 0, \dot{\phi}\right)_E$, where the dot stands for derivation with respect to the affine parameter τ along the geodesics. In the timelike case τ corresponds to the proper time. On the other hand, we suppose that the detector is static (i.e., O_∞'s four-velocity is tangent to the static Killing field $\partial/\partial t$), and in the chosen coordinate system its four-velocity is $u_\infty^\mu = \left(\dot{t}, 0, 0, 0\right)_\infty$ (Nucamendi et al 2001).

The motion of a test particle in the gravitational field on the brane can be described by the Lagrangian (Mak and Harko 2004)

$$2L = \left(\frac{ds}{d\tau}\right)^2 = -e^{\nu(r)}\left(\frac{dt}{d\tau}\right)^2 + e^{\lambda(r)}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\Omega}{d\tau}\right)^2, \tag{19}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. From the Lagrange equations it follows that we have two constants of motion, the energy $E = e^{\nu(r)}\dot{t}$ and the angular momentum $l = r^2\dot{\phi}$ (Lake 2004). The condition $u^\mu u_\mu = -1$ gives $-1 = -e^{\nu(r)}\dot{t}^2 + e^{\lambda(r)}\dot{r}^2 + r^2\dot{\phi}^2$ and, with the use of the constants of motion we obtain

$$E^2 = e^{\nu+\lambda}\dot{r}^2 + e^\nu\left(\frac{l^2}{r^2} + 1\right). \tag{20}$$

This equation shows that the radial motion of the particles on a geodesic is the same as that of a particle with position dependent mass and with energy $E^2/2$ in ordinary Newtonian mechanics moving in the effective potential $V_{eff}(r) = e^{\nu(r)}\left(l^2/r^2 + 1\right)$. The conditions for

circular orbits $\partial V_{eff}/\partial r = 0$ and $\dot{r} = 0$ lead to (Lake 2004)

$$l^2 = \frac{1}{2}\frac{r^3\nu'}{1-\frac{r\nu'}{2}}, E^2 = \frac{e^\nu}{1-\frac{r\nu'}{2}}. \tag{21}$$

The rotation curves of spiral galaxies are inferred from the red and blue shifts of the emitted radiation by stars moving in circular orbits on both sides of the central region. The light signal travels on null geodesics with tangent k^μ. We may restrict k^μ to lie in the equatorial plane $\theta = \pi/2$ and evaluate the frequency shift for a light signal emitted from O_E in circular orbit and detected by O_∞. The frequency shift associated to the emission and detection of the light signal is given by

$$z = 1 - \frac{\omega_E}{\omega_\infty}, \tag{22}$$

where $\omega_I = -k_\mu u_I^\mu$, and the index I refers to emission $(I = E)$ or detection $(I = \infty)$ at the corresponding space-time point (Nucamendi et al 2001; Lake 2004). Two frequency shifts corresponding to maximum and minimum values are associated with light propagation in the same and opposite direction of motion of the emitter, respectively. Such shifts are frequency shifts of a receding or approaching star, respectively. Using the constancy along the geodesic of the product of the Killing field $\partial/\partial t$ with a geodesic tangent gives the expressions of the two shifts as (Nucamendi et al 2001; Lake 2004)

$$z_\pm = 1 - e^{[\nu_\infty - \nu(r)]/2}\frac{1 \mp \sqrt{r\nu'/2}}{\sqrt{1 - r\nu'/2}}, \tag{23}$$

respectively, where $\exp\left[\nu\left(r\right)\right]$ represents the value of the metric potential at the radius of emission r and $\exp\left[\nu_\infty\right]$ represents the corresponding value of $\exp\left[\nu\left(r\right)\right]$ for $r \to \infty$. It is convenient to define two other quantities $z_D = \left(z_+ - z_-\right)/2$ and $z_A = \left(z_+ + z_-\right)/2$, given by

$$z_D\left(r\right) = e^{[\nu_\infty - \nu(r)]/2}\frac{\sqrt{r\nu'/2}}{\sqrt{1 - r\nu'/2}}, \tag{24}$$

$$z_A\left(r\right) = 1 - \frac{e^{[\nu_\infty - \nu(r)]/2}}{\sqrt{1 - r\nu'/2}}, \tag{25}$$

respectively, which can be easily connected to the observations (Nucamendi et al 2001). z_A and z_D satisfy the relation $\left(z_A - 1\right)^2 - z_D^2 = \exp\left[2\left(\nu_D - \nu\left(r\right)\right)\right]$, and thus in principle $\exp\left[\nu\left(r\right)\right]$ can be obtained directly from the observations. This could provide a direct observational test of the galactic geometry, and implicitly, of the brane world and other extra-dimensional models.

The line element on the brane, given by Eq. (12), can be rewritten in terms of the spatial components of the velocity, normalized with the speed of light, measured by an inertial observer far from the source, as $ds^2 = -dt^2\left(1 - v^2\right)$ (Mak and Harko 2004), where

$$v^2 = e^{-\nu}\left[e^{\lambda}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\Omega}{dt}\right)^2\right]. \tag{26}$$

For a stable circular orbit $\dot{r} = 0$, and the tangential velocity of the test particle can be expressed as

$$v_{tg}^2 = \frac{r^2}{e^{\nu}}\left(\frac{d\Omega}{dt}\right)^2. \tag{27}$$

In terms of the conserved quantities the angular velocity is given by

$$v_{tg}^2 = \frac{e^{\nu}}{r^2}\frac{l^2}{E^2}. \tag{28}$$

With the use of Eqs. (21) we obtain

$$v_{tg}^2 = \frac{r\nu'}{2}. \tag{29}$$

Thus, the rotational velocity of the test body is determined by the metric coefficient $\exp(\nu)$ only.

4. Galactic metric, dark radiation and dark pressure on the brane

The tangential velocity v_{tg} of stars moving like test particles around the center of a galaxy is not directly measurable, but can be inferred from the redshift z_{∞} observed at spatial infinity, for which $1 + z_{\infty} = \exp\left[\left(\nu_{\infty} - \nu\right)/2\right]\left(1 \pm v_{tg}\right)/\sqrt{1 - v_{tg}^2}$. Because of their non-relativistic velocities in galaxies bounded by $v_{tg} \leq (4/3) \times 10^{-3}$, we observe $v_{tg} \approx z_{\infty}$ (as the first part of a geometric series), with the consequence that the lapse $\exp(\nu)$ function necessarily tends to unity, i. e., $e^{\nu} \approx e^{\nu_{\infty}}/\left(1 - v_{tg}^2\right) \to 1$. The observations show that at distances large enough from the galactic center $v_{tg} \approx$ constant (Binney and Tremaine 1987; Persic et al. 1996; Boriello and Salucci 2001).

In the following we use this observational constraint to reconstruct the metric of a galaxy on the brane. The condition

$$v_{tg}^2 = \frac{1}{2}r\nu' = \text{constant}, \tag{30}$$

immediately allows to find the metric tensor component e^{ν} in the flat rotation curves region on the brane as

$$e^{\nu} = \left(\frac{r}{R_b}\right)^{2v_{tg}^2}, \tag{31}$$

where R_b is a constant of integration.

By adding Eqs. (14) and (15) and eliminating the dark radiation term U from the resulting equation and Eq. (13) gives the following equation satisfied by the unknown metric tensor components on the brane:

$$e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{2\nu'}{r} - \frac{2\lambda'}{r} - \frac{\nu'\lambda'}{2} + \frac{2}{r^2}\right) - \frac{2}{r^2} + 4\Lambda = 0. \tag{32}$$

Substituting ν given by Eq. (31) leads to a first order linear differential equation satisfied by $e^{-\lambda}$ and which is given by

$$\frac{d}{dr}e^{-\lambda} = -2\frac{v_{tg}^2\left(v_{tg}^2+1\right)+1}{v_{tg}^2+2}\frac{1}{r}e^{-\lambda} + \frac{2}{v_{tg}^2+2}\frac{1}{r} - \frac{4\Lambda}{v_{tg}^2+2}r. \tag{33}$$

From Eq. (33) we obtain $e^{-\lambda}$ as

$$e^{-\lambda} = \frac{1}{\left(v_{tg}^2+2\right)\alpha} + C_2 r^{-2\alpha} - \frac{2\Lambda}{\left(v_{tg}^2+2\right)(\alpha+1)}r^2, \tag{34}$$

where C_2 is an arbitrary constant of integration and we denoted

$$\alpha = \frac{v_{tg}^2\left(v_{tg}^2+1\right)+1}{v_{tg}^2+2}. \tag{35}$$

Eqs. (31) and (34) give the complete galactic metric on the brane.

Once the metric tensor components are known, the calculation of the dark radiation and dark pressure terms is straightforward. Eq. (13) immediately gives the dark radiation U on the brane as

$$\frac{48\pi G}{k_4^4\lambda_b}U(r) = \frac{1}{v_{tg}^2+2}\left[\frac{2(2-\alpha)}{(\alpha+1)} - v_{tg}^2\right]\Lambda + \left[1 - \frac{1}{\left(v_{tg}^2+2\right)\alpha}\right]\frac{1}{r^2} + (2\alpha-1)\,C_2 r^{-2\alpha-2}. \tag{36}$$

From Eqs. (13) and (14) we obtain the dark pressure on the brane as

$$\frac{96\pi G}{k_4^4\lambda_b}P(r) = e^{-\lambda}\left(\frac{3\nu'}{r} + \frac{4}{r^2} - \frac{\lambda'}{r}\right) - \frac{4}{r^2} + 4\Lambda, \tag{37}$$

or

$$\frac{96\pi G}{k_4^4\lambda_b}P(r) = \frac{4}{v_{tg}^2+2}\left[v_{tg}^2+1-\frac{2\left(v_{tg}^2+2\right)\alpha+5v_{tg}^2+1}{\left(v_{tg}^2+2\right)(\alpha+1)}\right]\Lambda+\frac{2}{v_{tg}^2+2}\left[\frac{5v_{tg}^2+1}{\left(v_{tg}^2+2\right)\alpha}-2v_{tg}^2-1\right]\frac{1}{r^2}$$
$$+2C_2\frac{2\left(v_{tg}^2+2\right)\alpha+5v_{tg}^2+1}{v_{tg}^2+2}r^{-2\alpha-2}. \tag{38}$$

The angular momentum and the energy of a star moving in the galactic gravitational field on the brane are given by

$$l = r\frac{v_{tg}}{\sqrt{1-v_{tg}^2}}, \tag{39}$$

and

$$E = \left(\frac{r}{R_b}\right)^{v_{tg}^2}\frac{1}{\sqrt{1-v_{tg}^2}}, \tag{40}$$

respectively. For the effective potential describing the Newtonian motion of a test particle we find

$$V_{eff}(r) = \left(\frac{r}{R_b}\right)^{2v_{tg}^2}\frac{1}{1-v_{tg}^2}. \tag{41}$$

From Eq. (18) it follows that the dark mass associated to the dark radiation term is given by

$$2GM_U(r) = \frac{1}{3\left(v_{tg}^2+2\right)}\left[\frac{2(2-\alpha)}{(\alpha+1)}-v_{tg}^2\right]\Lambda r^3+\left[1-\frac{1}{\left(v_{tg}^2+2\right)\alpha}\right]r-C_2r^{-2\alpha-1}. \tag{42}$$

At distances relatively close to the galactic center, where the effect of the cosmological constant can be neglected, but sufficiently high so that the last term in Eq. (42) is also negligible small, the dark mass can be approximated as

$$2GM_U(r) \approx \frac{v_{tg}^2\left(v_{tg}^2+1\right)}{v_{tg}^2+2}r \approx \frac{v_{tg}^2}{2}r. \tag{43}$$

The dark mass is linearly increasing with the radial distance from the galactic center.

The behavior of the metric coefficients and of the dark radiation and pressure in the solutions we have obtained depends on two arbitrary constants of integration R_b and C_2. Their numerical value can be obtained by assuming the continuity of the metric coefficient $\exp(\lambda)$ across the vacuum boundary of the galaxy. For simplicity we assume that inside the

"normal" (baryonic) luminous matter, with density ρ_B, which form a galaxy, the non-local effects of the Weyl tensor can be neglected. We define the vacuum boundary r_0 of the galaxy (which for simplicity is assumed to have spherical symmetry) by the condition $\rho_B(r_0) \approx 0$.

Therefore at the vacuum boundary of the galaxy the metric coefficients are $\exp(\nu) = 1 - 2GM_B/r_0$ and $\exp(-\lambda) = 1 - 2GM_B/r_0$, where $M_B = 4\pi \int_0^{r_0} \rho_B(r) r^2 dr$ is the total baryonic mass inside the radius r_0. The continuity of $\exp(\nu)$ through the surface $r = r_0$ gives

$$R_b = r_0 \left(1 - \frac{2GM_B}{r_0}\right)^{-\frac{1}{2}v_{tg}^{-2}}, \tag{44}$$

while the continuity of $\exp(-\lambda)$ fixes the integration constant C_2 as

$$C_2 = \frac{r_0^{2\alpha}}{v_{tg}^2 + 2}\left[\left(1 - \frac{2GM_B}{r_0}\right)\left(v_{tg}^2 + 2\right) - \frac{1}{\alpha} + \frac{2\Lambda}{\alpha + 1} r_0^2\right]. \tag{45}$$

Thus at the galactic level the metric coefficients and the dark radiation and pressure on the brane can be obtained in terms of observable quantities.

5. Light deflection and lensing by galaxies in brane world models

One of the ways we could in principle test the galactic metric obtained in the previous Section would be by studying the light deflection by galaxies, and in particular by studying the deflection of photons passing through the region where the rotation curves are flat. Let us consider a photon approaching a galaxy from far distances. We will compute the deflection by assuming that the metric is given by Eqs. (31) and (34).

The bending of light on the brane by the galactic gravitational field results in a deflection angle $\Delta\phi$ given by

$$(\Delta\phi)_{brane} = 2\left|\phi(r_0) - \phi_\infty\right| - \pi, \tag{46}$$

where ϕ_∞ is the incident direction and r_0 is the coordinate radius of the closest approach to the center of the galaxy, and generally (Nucamendi et al 2001)

$$\phi(r_0) - \phi_\infty = \int_{r_0}^{\infty} e^{\frac{\lambda(r)}{2}} \left[e^{\nu(r_0) - \nu(r)}\left(\frac{r}{r_0}\right)^2 - 1\right]^{-1/2} \frac{dr}{r}. \tag{47}$$

By taking into account the explicit expressions of the metric tensor components in the

flat rotation curves region we obtain

$$\phi\left(r_0\right)-\phi_{\infty}=\int_{r_0}^{\infty}\frac{dr}{r\sqrt{\left[\frac{1}{\left(v_{tg}^2+2\right)\alpha}+C_2r^{-2\alpha}-\frac{2\Lambda}{\left(v_{tg}^2+2\right)(\alpha+1)}r^2\right]\left[e^{\nu(r_0)}r_0^{-2}R_b^{2v_{tg}^2}r^{2\left(1-v_{tg}^2\right)}-1\right]}}. \tag{48}$$

By introducing a new variable $\eta = r/r_0$ and taking into account the matching condition given by Eq. (65) we can rewrite Eq. (48) as

$$\phi\left(r_0\right)-\phi_{\infty}=\int_{1}^{\infty}\frac{\sqrt{v_{tg}^2+2}\eta^{-1}\left[\eta^{2\left(1-v_{tg}^2\right)}-1\right]^{-1/2}d\eta}{\sqrt{\left[\frac{1}{\alpha}+\eta^{-2\alpha}\left[\left(1-\frac{2GM_B}{r_0}\right)\left(v_{tg}^2+2\right)-\frac{1}{\alpha}+\frac{2\Lambda}{\alpha+1}r_0^2\right]-\frac{2\Lambda r_0^2}{\alpha+1}\eta^2\right]}}. \tag{49}$$

In standard general relativity the bending angle by a galaxy is given by $(\Delta\phi)_{GR} = 2\left|\phi\left(r_0\right)-\phi_{\infty}\right|-\pi = 4GM_B/r_0$. The total deflection angle is given by $(\Delta\phi)_{total} = (\Delta\phi)_{brane} + (\Delta\phi)_{GR}$.

In order to estimate the magnitude of the bulk contribution to the bending of light on the brane and to compare this contribution to the results from standard dark matter models we define the parameter

$$\delta = \frac{(\Delta\phi)_{brane}}{(\Delta\phi)_{DM}}, \tag{50}$$

where $(\Delta\phi)_{DM}$ is the deflection angle as is usually considered in the standard dark matter models. Generally, in the dark halo models, a light ray which goes past the halo without entering it propagates entirely in a Schwarzschild metric and the light ray is deflected by an angle $(\Delta\phi)_{GR}$. The bending of the light which passes through the dark matter halo is determined by the metric inside the halo and this depends on the assumed properties of the dark matter.

We shall compute the parameter δ in a semi-realistic model for dark matter, in which it is assumed that the galaxy (the baryonic matter) is embedded into an isothermal mass distribution (the dark matter), with the density varying as $\rho = \sigma_v^2/2\pi G r^2$, where σ_v is the line of sight velocity dispersion (Binney and Tremaine 1987). In this model it is assumed that the mass distribution of the dark matter is spherically symmetric. In fact, if the rotation curve is flat, then the mass distribution must be that of the isothermal sphere, for which we also have $v_{tg} = \sqrt{2}\sigma_v$. The surface density Σ of the isothermal sphere is $\Sigma\left(r\right) = \sigma_v^2/2Gr$. For this dark matter distribution the bending angle of light is constant and is given by $(\Delta\phi)_{DM} = 2\pi v_{tg}^2$ (Blanford and Narayan 1992).

Therefore in this model

$$\delta = \frac{(\Delta\phi)_{brane}}{2\pi v_{tg}^2}. \tag{51}$$

The variation of the parameter δ as a function of the quantity $\chi = 4GM_B/r_0$ is represented in Fig. 1.

In order to find an approximate analytic expression for the deflection of light due to the presence of the Weyl stresses on the brane we perform a Taylor series expansion of the function $F(\eta) = \left(1/\left(v_{tg}^2+2\right)\alpha + C_2\eta^{-2\alpha} - 2\Lambda r_0^2\eta^2/\left(v_{tg}^2+2\right)(\alpha+1)\right)^{-1/2}$ near $\eta = 1$. In the first order we obtain

$$F(\eta) \approx \left[\frac{1}{\left(v_{tg}^2+2\right)\alpha} + C_2 r_0^{-2\alpha} - \frac{2\Lambda r_0^2}{\left(v_{tg}^2+2\right)(\alpha+1)}\right]^{-1/2} - \frac{2\left(C_2 r_0^{-2\alpha}\alpha + \frac{2\Lambda r_0^2}{\left(v_{tg}^2+2\right)(\alpha+1)}\right)(\eta-1)}{\left[\frac{1}{\left(v_{tg}^2+2\right)\alpha} + C_2 r_0^{-2\alpha} - \frac{2\Lambda r_0^2}{\left(v_{tg}^2+2\right)(\alpha+1)}\right]^{3/2}} + ... \tag{52}$$

Substituting this expression into Eq. (49) gives, after evaluating the integral, the first order gravitational deflection angle of the light on the brane as

$$\begin{aligned}(\Delta\phi)_{brane} &= 2\left|\frac{\pi\sqrt{\frac{1}{\left(v_{tg}^2+2\right)\alpha} + C_2 - \frac{2\Lambda r_0^2}{\left(v_{tg}^2+2\right)(\alpha+1)}}}{2\left(1-v_{tg}^2\right)} - \right. \\ &\left. \frac{\sqrt{\pi}\left(C_2\alpha + \frac{2\Lambda r_0^2}{\left(v_{tg}^2+2\right)(\alpha+1)}\right)\left[\frac{\sqrt{\pi}}{2\left(1-v_{tg}^2\right)} + \frac{\Gamma\left(-\frac{1}{2}\frac{v_{tg}^2}{1-v_{tg}^2}\right)}{\Gamma\left(-\frac{1}{2}\frac{1}{1-v_{tg}^2}\right)}\right]}{\left(\frac{1}{\left(v_{tg}^2+2\right)\alpha} + C_2 - \frac{2\Lambda r_0^2}{\left(v_{tg}^2+2\right)(\alpha+1)}\right)^{3/2}}\right| - \pi,\end{aligned} \tag{53}$$

where $\Gamma(z)$ is the Euler gamma function, defined as $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

Alternatively, some very simple expressions for the parameter δ can be derived in the limits $\chi >> v_{tg}^2$ and $\chi << v_{tg}^2$, respectively. By assuming that $v_{tg}^2 << 1$ and neglecting all terms containing the tangential velocity and the cosmological constant, the light deflection angle on the brane becomes

$$(\Delta\phi)_{brane} \approx 2\left|\int_1^\infty \eta^{-1}\left[\left(\eta^2-1\right)\left(2-\chi\eta^{-1}\right)\right]^{-1/2}d\eta\right| - \pi. \tag{54}$$

The integral can be calculated exactly, and we obtain

$$(\Delta\phi)_{brane} \approx \left| \frac{\sqrt{2}}{4} \frac{4\sqrt{2}K\left(\frac{2\chi}{\chi-2}\right) +_3 F_2\left[\left\{\frac{3}{4},1,\frac{5}{4}\right\},\left\{\frac{3}{2},\frac{3}{2}\right\},\frac{\chi^2}{4}\right]\sqrt{2-\chi}\chi}{\left(1-\frac{\chi}{2}\right)^{1/2}} \right| - \pi, \tag{55}$$

where $K(m)$ is the complete elliptic integral of the first kind given by $K(m) = \int_0^{\pi/2}\left(1-m\sin^2\theta\right)^{-1/2} d\theta$ and ${}_pF_q\left(\vec{a};\vec{b};z\right)$ is the generalized hypergeometric function defined as

$${}_pF_q\left(\vec{a};\vec{b};z\right) = \sum_{k=0}^{\infty} (a_1)_k \ldots (a_p)_k \,/\, (b_1)_k \ldots (b_q)_k \, z^k/k!. \tag{56}$$

A series expansion of Eq. (55) gives

$$(\Delta\phi)_{brane} \approx \frac{\chi}{2}. \tag{57}$$

Therefore for the isothermal dark matter model and in the limit of large χ the parameter δ tends to the value $\chi/4\pi v_{tg}^2$, $\delta \to \chi/4\pi v_{tg}^2$.

Neglecting χ and the cosmological constant in Eq. (49) gives

$$(\Delta\phi)_{brane} \approx 2\left|\int_1^{\infty} \eta^{-1}\left[\left(\eta^2-1\right)\left(2+v_{tg}^2\eta^{-1}\right)\right]^{-1/2} d\eta\right| - \pi. \tag{58}$$

A similar calculation as above gives $(\Delta\phi)_{brane} \approx v_{tg}^2/2$. Hence in the limit of small χ, for the isothermal dark matter model δ can be approximated as $\delta \approx 1/4\pi$. These two limiting forms of the parameter δ are consistent with the numerical results presented in Fig. 1.

If the effect of the cosmological constant can be neglected, the deflection angle of the light in the Weyl stresses dominated region around a galaxy is a function of the tangential velocity of the test particles in stable circular orbit only. To obtain the total value of the bending angle of the light one must also add the usual general relativistic contribution to the bending from the baryonic mass of the galaxy.

Once the light deflection angle is known, one can study the gravitational lensing on the brane in the flat rotation curves region. The lensing geometry is illustrated in Fig. 2.

The light emitted by the source S is deflected by the lens L (a galaxy in our case) and reaches the observer O at an angle θ to the optic axis OL, instead of β. The lens L is located at a distance D_L to the observer and a distance D_{LS} to the source, respectively, while the

observer-source distance is D_S. r_0 is the impact factor (distance of closest approach) of the photon beam.

The lens equation is given by (Whisker 2005)

$$\tan\beta = \tan\theta - \frac{D_{LS}}{D_S}\left[\tan\theta + \tan\left(\Delta\phi - \theta\right)\right]. \tag{59}$$

By assuming that the angle θ is small, we have $\tan\theta \approx \theta$ and the lens equation can be written as

$$\beta \approx \theta - \frac{D_{LS}}{D_S}\Delta\phi. \tag{60}$$

In the special case of the perfect alignment of the source, lens and observers, $\beta = 0$, and the azimuthal axial symmetry of the problem yields a ring image, the Einstein ring, with angular radius

$$\theta_E^{(brane)} \approx \frac{D_{LS}}{D_S}\Delta\phi. \tag{61}$$

This equation can be expressed in a more familiar form by taking into account that the impact parameter $r_0 \approx D_L\theta$, which gives

$$\theta_E^{(brane)} \approx \sqrt{\frac{D_{LS}}{D_S D_L}\Delta\phi r_0} \approx \theta_E^{(GR)}\sqrt{\frac{r_0}{GM_B}}\sqrt{\Delta\phi}, \tag{62}$$

where $\theta_E^{(GR)}$ is the angular radius of the Einstein ring in the case of standard general relativity, $\theta_E^{(GR)} = \sqrt{4\left(D_{LS}/D_S D_L\right)GM_B}$.

In the case of a galaxy with a heavy isothermal dark matter distribution, the Einstein radius of the lens formed in perfect alignment is (Blanford and Narayan 1992)

$$\theta_E^{(DM)} = \left(\frac{4\pi\sigma_v^2}{c^2}\right)\frac{D_{LS}}{D_S}. \tag{63}$$

The ratio δ of the Einstein's rings angular diameters in the brane world models and in the isothermal dark galactic halo model is

$$\delta = \frac{\theta_E^{(brane)}}{\theta_E^{(DM)}} = \frac{(\Delta\phi)_{brane}}{2\pi v_{tg}^2}. \tag{64}$$

Therefore the ratio of the angular radii of the Einstein rings in the brane world models and in the isothermal dark matter model is given by the same parameter δ which has been already introduced in Eq. (50). The variation of the ratio of the Einstein rings in the two models is presented in Fig. 1.

6. Discussions and final remarks

The galactic rotation curves continue to pose a challenge to present day physics as one would to have a better understanding of some of the intriguing phenomena associated with them, like their universality and the very good correlation between the amount of dark matter and the luminous matter in the galaxy. To explain these observations models based on particle physics in the framework of Newtonian gravity are the most commonly considered.

In the present paper we have considered and further developed an alternative view to the dark matter problem, namely, that the galactic rotation curves can naturally be explained in models in which our Universe is a domain wall (a brane) in a multi-dimensional space-time (the bulk). The extra-terms in the gravitational field equations on the brane induce a supplementary gravitational interaction, which can account for the observed behavior of the galactic rotation curves. By using the simple observational fact of the constancy of the galactic rotation curves, the galactic metric and the corresponding Weyl stresses (dark radiation and dark pressure) can be completely reconstructed.

The form of the galactic metric we have obtained in the framework of the brane world models differs from what would be naively expected, that is, $ds^2 = -(1+2\Phi)\,dt^2 + (1+2\Phi)^{-1}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, with Φ representing the Newtonian potential. This form is often implicitly assumed (Pal et al. 2005), and the fact that it is not appropriate for the region where the rotation curves are flat can lead to significant errors in the estimation of the magnitude of some important physical effects, like, for example, the bending of light by the galaxies.

The observations in spiral galaxies usually determine v_{tg} from the redshift z_D, so that $z_D \approx v_{tg} \approx$ constant. By assuming that $\exp(\nu_\infty) \approx 1$, and with the use of Eqs. (24) and (31), it follows that the results we have obtained are self-consistent if the condition

$$\left(\frac{r}{R_b}\right)^{-v_{tg}^2}\left(1-v_{tg}^2\right)^{-1/2} \approx 1, \tag{65}$$

holds. By using the approximation $(r/R_b)^{-v_{tg}^2} \approx 1 - v_{tg}^2 \ln(r/R_b)$, valid for $\left|v_{tg}^2 \ln(r/R_b)\right| << 1$, it follows that the condition is satisfied if $\left|\ln(r/R_b)\right| << v_{tg}^{-2}$. Since $v_{tg} \approx 10^{-3} - 10^{-4}$, the approximations we have used are self-consistent as long as $-10^6 << \ln(r/R_b) << 10^6$, condition which, for the case of the galaxies, does not impose any practically relevant constraint.

As a second consistency condition we require that the timelike circular geodesics on the brane be stable. Let r_0 be a circular orbit and consider a perturbation of it of the form $r = r_0 + \delta$, where $\delta << r_0$ (Lake 2004). Taking expansions of $V_{eff}(r)$ and $\exp(\nu+\lambda)$ about

$r = r_0$, it follows from Eq. (20) that

$$\ddot{\delta} + \frac{1}{2} e^{\nu(r_0)+\lambda(r_0)} V''_{eff}(r_0)\, \delta = 0. \tag{66}$$

The condition for stability of the simple circular orbits requires $V''_{eff}(r_0) > 0$ (Lake 2004). This gives

$$2\left(\frac{r_0}{R_b}\right)^{2v_{tg}^2} r_0^{-4}\left[-v_{tg}^2\left(1-2v_{tg}^2\right) r_0^2 + l^2\left(3-5v_{tg}^2+2v_{tg}^4\right)\right] > 0. \tag{67}$$

By neglecting the small terms containing powers of the tangential velocity v_{tg} with respect to the unity, it follows that the circular orbits are stable if their radius satisfy the condition $r_0^2 < 3l^2/v_{tg}^2$.

In the present model there is a very simple relation between the mass of the inter-galactic dark radiation M_U and the luminous (baryonic) mass M_B of the galaxy. By assuming that the tangential velocity of particles in circular orbit is approximately given by $v_{tg}^2 \approx GM_B/r_0$, it follows that M_U is related to M_B via the following simple scaling relation

$$M_U(r) \approx \frac{r}{2r_0} M_B. \tag{68}$$

The mass of the dark radiation is proportional to the mass of the galaxy and is linearly increasing with the distance to the galactic center.

If we assume that the flat rotation curves extend indefinitely, the resulting space-time is not asymptotically flat, but of de Sitter type. This is due to the presence of the cosmological constant Λ on the brane. Observationally, the galactic rotation curves remain flat to the farthest distances that can be observed. On the other hand there is a simple way to estimate an upper bound for the cutoff of the constancy of the tangential velocities. The idea is to consider the point at which the decaying density profile of the dark radiation associated to the galaxy becomes smaller than the average energy density of the Universe. Let the value of the coordinate radius at the point where the two densities are equal be $R_U^{\max}$. Then at this point $\left(8\pi G/k_4^4\lambda_b\right) U\left(R_U^{\max}\right) = \left(8\pi G/c^2\right)\rho_{univ}$, where $\rho_{univ}c^2$ is the mean energy density of the universe. Hence we obtain

$$R_U^{\max} = \sqrt{\frac{1}{\frac{8\pi G}{c^2}\rho_{univ} - \frac{1}{v_{tg}^2+2}\left[\frac{2(2-\alpha)}{(\alpha+1)} - v_{tg}^2\right]\Lambda}}\,\frac{\sqrt{v_{tg}^2+1}}{\sqrt{v_{tg}^2\left(v_{tg}^2+1\right)+1}}\, v_{tg}. \tag{69}$$

The mean density of the universe and the value of the cosmological constant can be expressed with the help of the density parameters $\Omega = \rho_{univ}/\rho_{crit}$ and $\Omega_\Lambda = \Lambda c^2/3H_0^2$,

respectively, where $\rho_{crit} = 3H_0^2/8\pi G$, with H_0 is the Hubble constant, given by $H_0 = 100h$ km/sec Mpc, $1/2 \leq h \leq 1$. Therefore

$$R_U^{\max} = \frac{c}{\sqrt{3}} H_0^{-1} \sqrt{\frac{1}{\Omega - \frac{1}{v_{tg}^2+2}\left[\frac{2(2-\alpha)}{(\alpha+1)} - v_{tg}^2\right]\Omega_\Lambda} \frac{\sqrt{v_{tg}^2+1}}{\sqrt{v_{tg}^2\left(v_{tg}^2+1\right)+1}}} v_{tg}. \tag{70}$$

A numerical evaluation of $R_U^{\max}$ requires the knowledge of the ratio of the rotation velocities in the flat region and of the basic fundamental cosmological parameters. For v_{tg} in the range $v_{tg} \in (10^{-4}, 10^{-3})$ and for $\Omega = 1$, $\Omega_\Lambda = 0.7$ we obtain $R_U^{\max} \in (315h^{-1}, 3150h^{-1})$ kpc. On the other hand the measured flat regions are about $R \approx 2 \times R_{opt}$, where R_{opt} is the radius encompassing 83% of the total integrated light of the galaxy (Binney and Tremaine 1987; Persic et al. 1996; Boriello and Salucci 2001). If we take as a typical value $R \approx 30$ kpc, then it follows that $R < R_U^{\max}$. However, according to our model, the flat rotation curves region should extend far beyond the present measured range.

An alternative estimation of $R_U^{\max}$ can be obtained from the observational requirement that at the cosmological level the energy density of the dark matter represents a fraction $\Omega_m \approx 0.3$ of the total energy density of the universe $\Omega = 1$. Therefore the dark matter contribution inside a radius $R_U^{\max}$ is given by $4\pi\Omega_m \left(R_U^{\max}\right)^3 \rho_{crit}/3$, which gives

$$R_U^{\max} \approx \sqrt{\frac{1}{2\Omega_m}} \frac{c}{H_0} v_{tg}. \tag{71}$$

Therefore, by assuming that the dark radiation contribution to the total energy density of the Universe is of the order of $\Omega_m \approx 0.3$ we have $R_U^{\max} \in (388h^{-1}, 3881h^{-1})$ kpc for $v_{tg} \in (10^{-4}, 10^{-3})$.

The limiting radius at which the effects of the extra-dimensions extend, far away from the baryonic matter distribution, is given in the present model by Eq. (70). In the standard dark matter models this radius is called the truncation parameter s, and it describes the extent of the dark matter halos. Values of the truncation parameter by weak lensing have been obtained for several fiducial galaxies by Hoekstra et al. (2004). In the following we compare our results with the observational values of s obtained by fitting the observed values with the truncated isothermal sphere model, as discussed in some detail in Hoekstra et al. (2004). The truncation parameter s is related to R_U by the relation $s = R_U/2\pi$ (see Eq. (4) in Hoekstra et al. (2004) and Eq. (43) in the present paper). Therefore, generally s can be obtained from the relation

$$s \approx \frac{\sigma}{\sqrt{6}\pi} \sqrt{\frac{1}{2\Omega_m}} H_0^{-1}, \tag{72}$$

where σ is the velocity dispersion, expressed in km/s. Hence the truncation parameter is a simple function of the velocity dispersion and of the cosmological parameters only. For a velocity dispersion of $\sigma = 146$ km/s and with $\Omega_m = 0.3$, Eq. (72) gives $s \approx 245h^{-1}$ kpc, while the truncation size obtained observationally in Hoekstra et al. (2004) is $s = 213h^{-1}$ kpc. For $\sigma = 110$ km/s we obtain $s \approx 184h^{-1}$ kpc, while $\sigma = 136$ km/s gives $s \approx 228h^{-1}$ kpc. All these values are consistent with the observational results reported in Hoekstra et al. (2004), the error between prediction and observation being of the order of 20%. We have also to mention that the observational values of the truncation parameter depend on the scaling relation between the velocity dispersion and the fiducial luminosity of the galaxy. Two cases have been considered in Hoekstra et al. (2004), the case in which the luminosity L_B does not evolve with the redshift z and the case in which L_B scales with z as $L_B \propto (1+z)$. Depending on the scaling relation slightly different values of the velocity dispersion and truncation parameter are obtained.

The study of the deflection of light (gravitational lensing) in the flat rotation curves region can provide a powerful observational tool for discriminating between standard dark matter and brane world models. Due to the fixed form of the galactic metric on the brane, the light bending angle is a function of the tangential velocity of particles in stable circular orbit and the baryonic mass and radius of the galaxy. The specific form of the bending angle is determined by the brane galactic metric, and this form is very different as compared to the other dark matter models (long-range self-interacting scalar fields, MOND, non-symmetric gravity etc.). When $\chi = 4GM_B/r_0 << 1$, the gravitational light deflection angle is much larger than the value predicted by the standard general relativistic approach. Even when we compare our results with standard dark matter models, like the isothermal dark matter halo model, we still find significant differences in the lensing effect. Therefore the study of the gravitational lensing may provide evidence for the existence of the bulk effects on the brane. On the other hand, since in this model there is only baryonic matter, all the physical properties at the galactic level are determined by the amount of the luminous matter and its distribution.

Generally, the angular position of the source β is related to the angular position of its image by the lens equation, $\beta = \theta + \alpha(\theta)$. The standard general relativistic deflection law for a point mass in general relativity is $\alpha(\theta) = -\theta_E^2/\theta$ Mortlock and Turner (2001). In order to test alternative theories of gravity a more general point mass deflection law, of the form $\alpha(\theta) = -\left(\theta_E^2/\theta\right)\left(\theta_0/\theta + \theta_0\right)^{\xi-1}$, $\xi \geq 1$, was introduced by Mortlock and Turner (2001) (see Hoekstra et al. (2002) for a more general parametrization). Here θ_0 is a parameter which can be related to the scale r_0 beyond which the physics becomes non-Newtonian. In the brane world model the function α can be represented as $\alpha(\theta) = -\left[\left(\theta_E^{(GR)}\right)^2/\theta\right]\left(r_0/GM_B\right)\left(\Delta\phi\right)_{brane}$. From the

deflection law one can find the tangential shear of the image, $\gamma_{\tan}(\theta) = \left[(d\alpha/d\theta) - \alpha(\theta)/\theta\right]/2$. For the brane world model we obtain

$$\gamma_{\tan}(\theta) = \frac{\left(\theta_E^{(GR)}\right)^2}{\theta^2}\frac{r_0}{GM_B}(\Delta\phi)_{brane}. \tag{73}$$

By using the definition of the parameter δ we obtain $(\Delta\phi)_{brane} = \delta(\Delta\phi)_{DM} = 2\pi\delta v_{tg}^2$, where, for simplicity, we have adopted again the isothermal sphere model for "standard " dark matter. Moreover, we assume that the tangential velocity is related to the baryonic mass by the (approximate) Keplerian value, $v_{tg}^2 \approx GM_B/r_0$. Therefore we obtain for the tangential shear the following simple expression

$$\gamma_{\tan}(\theta) \approx 2\pi\delta\frac{\left(\theta_E^{(GR)}\right)^2}{\theta^2}. \tag{74}$$

This form of the tangential shear is similar to that one resulting from a modification of the deflection law proposed by Hoekstra et al. (2002), and which is given by $\gamma_{\tan}(\theta) = (\theta_{out}/\theta_0)(\theta_E^2/\theta^2)$, where θ_{out} and θ_0 are some ad hoc parameters. The allowable range for the values of θ_0, obtained from observations, excludes small values. The brane world model fixes the values of these parameters as $\theta_{out}/\theta_0 = 2\pi\delta$. Since the ratio θ_{out}/θ_0 is a measure of the "mass discrepancy", δ could also be interpreted as a measure of the extra mass generated by the dark radiation. The best fit with the observational data is obtained for $\theta_{out} = s$, where s is the truncation parameter. Therefore δ can also be expressed as $\delta = R_U/\theta_0$.

The mean shear signal can be measured in galaxy-galaxy lensing observations. We compare the predictions of the brane world model, given by Eq. (74) with the observational data obtained in Hoekstra et al. (2004). By fitting an isothermal sphere model to the tangential shear at radii smaller than $2'$ (corresponding to a maximum radius of $\approx 350h^{-1}$ kpc) the mean value for the Einstein radius was found to be $\langle\theta_E\rangle = 0.''140 \pm 0.''009$. For this value of the Einstein radius the variation of $\gamma_{\tan}$ calculated for different values of the parameter δ is compared with the observational data presented in Hoekstra et al. (2004), in Fig. 3. The best fit with the observational data corresponds to large values of δ, $\delta \approx 8 - 12$. Since the velocity dispersion for the considered sample of galaxies is around $\sigma \in (128, 150)$ km/s, corresponding to $v_{tg} \approx 150 - 210$ km/s, it follows that the ratio of the baryonic mass and the radius for these galaxies should be of the order of $\chi \in (1.5 \times 10^{-5}, 5 \times 10^{-5})$. If the baryonic mass-radius ratio for the galaxies is significantly smaller than this value, then the predictions of this simple brane world model could not fit satisfactorily the existing observational data. An estimate of the galactic mass-radius ratio can be obtained by approximating the tangential value by its Keplerian value, which gives $\chi \approx 8\sigma^2$. For $\sigma = 150$ km/s we

obtain $\chi \approx 2 \times 10^{-6}$. This simple estimate seems to indicate at first sight that the present observational data do not favor the brane world interpretation for dark matter. However, we have to mention that the estimation of the ratio of the baryonic mass and the radius from the tangential velocity could be affected by significant errors. It would be more realistic to assume that $\chi \approx 4k^2 v_{tg}^2 = 8k^2\sigma^2$, where k is a factor describing the uncertainty in the determination of the ratio of the "normal" mass and radius of a galaxy from the tangential velocity alone. If this factor is of the order of 2, $k \approx 2$, then the brane world model prediction could become consistent with the observations. Hence to find a definite answer to the possibility of the brane world models to correctly describe the weak lensing data a much more careful analysis of data is needed.

Therefore the study of brane world model galaxy-galaxy lensing and of the dark matter halos properties could provide strong constraints on the brane world model and on related high energy physics models.

The measurement of the azimuthally averaged tangential shear around galaxies is robust against residual systematics, that is, contributions from a constant or gradient residual shear cancel. However, this is not the case for the quadrupole signal (Hoekstra et al. 2004). If the lens galaxies are oriented randomly with respect to the residual shear, then the average over many lenses will cancel the contribution from systematics. But in real observations the uncorrected shapes of the lenses are aligned with the systematic signal. An imperfect correction can give rise to a small quadrupole signal, which can mimic a cosmic shear signal. Generally, the residual shear has an amplitude $\hat{\gamma}$ and is located with respect to the major axis of the lens at an angle ϕ. The tangential shear $\gamma_{\tan}^{obs}$ observed at a point (r, θ) is the sum of the lensing signal $\gamma_{\tan}^{lens}$ and the contribution from the systematics, $\hat{\gamma}_{\tan}$, so that $\gamma_{\tan}^{obs} = \gamma_{\tan}^{lens} + \hat{\gamma}_{\tan}$. $\hat{\gamma}_{\tan}$ is given by $\hat{\gamma}_{\tan} = -\hat{\gamma}\cos\left[2\left(\theta - \phi\right)\right]$ (Hoekstra et al. 2004). One way to estimate the flattening of the halo is to measure the shears $\gamma_{\tan}^{(+)}$ at $\theta = 0$ and π and $\gamma_{\tan}^{(-)}$ at $\theta = \pi/2$ and $3\pi/2$, respectively. The observed ratio is $f_{obs} = \gamma_{\tan}^{(-)}/\gamma_{\tan}^{(+)}$ can be written as $f_{obs} = \left[\gamma_{\tan}^{(-)} + \hat{\gamma}\cos\left(2\phi\right)\right] / \left[\gamma_{\tan}^{(+)} - \hat{\gamma}\cos\left(2\phi\right)\right]$ (Hoekstra et al. 2004). In the framework of the simple brane world model we have considered all the shear parameters (tangential as well as residual) are proportional to the values corresponding to the standard dark matter case. On the other hand the mean value of $\cos\left(2\phi\right)$ is proportional to the measure α of the correlation between the position angle of the lens and the direction of the systematic shear of the background galaxies, which is a very small quantity. The brane world model correction to α cannot lead to a significant increase to this quantity, due to the large separation distances between galaxies. Therefore we expect that f_{obs} is an invariant and brane world effects do not affect the robustness of the measurement of the average halo shape.

With the use of the approximate relation $\left(1 + x/n\right)^n \approx \exp\left(x\right)$, which is valid for large

n, we can write Eq. (44) in the form

$$R_b \approx r_0 \exp\left(\chi/4v_{tg}^2\right). \tag{75}$$

In the case of a very massive object with $\chi \approx 1$ and for $v_{tg} \in (10^{-4}, 10^{-3})$ it follows that $R_b >> r_0$. But for very small values of χ, of the same order as v_{tg}^2, $R_b \approx r_0$. Therefore flat rotation curves are specific for particles moving in stable circular orbits around galaxies (having $\chi << 1$), while the same phenomenon cannot be detected for very compact (stellar or black hole type) objects.

The measurement of the anisotropy in the lensing signal around galaxies and the detection of the flattening of the dark matter halos could pose some serious challenges to the alternative theories of gravitation (Hoekstra et al. 2002, 2004). In many theories proposing modifications of the dynamic law for gravitation the lensing signal caused by the intrinsic shapes of the galaxies decreases as $\propto r^{-2}$ and hence it is negligible at large distances. Therefore dark halos cannot be modelled as spherical systems. This important observational result can clearly discriminate between the gravitational explanations proposed as substitutes for dark matter. However, in the framework of the model discussed in the present paper, the flattening of the dark matter halos and the anisotropic signal can be easily explained, at least at a simple qualitative level. We have obtained all our results under the (unrealistic) assumption of the spherical symmetry, with the galaxy (consisting of baryonic matter only) modelled as a self-gravitating relativistic sphere. A much more realistic model for a galaxy is its representation as a self-gravitating rotating stationary axisymmetric disc, in which generally the metric coefficients depend on both polar coordinates r and θ. The brane world metric outside this baryonic matter distribution is also anisotropic, and we would naturally expect a flattening of the region in which the bulk effects are important, associated with an anisotropic (angular dependent) weak lensing signal. The "dark matter" is, in this model, a projection of the geometry of the higher dimensional space time far away from the baryonic matter distribution, which has to match the geometry of the galaxy. Therefore disc type or spheroidal distributions will automatically create flattened and anisotropic (non-spherical) geometrical effects. For example, already in standard general relativity in the weak field limit a cold disc has the associated gravitational potential $-\nu = \Phi = -v_{tg}^2 \ln\left[r\left(1 + |\cos\theta|\right)/D\right]$, where D is a fiducial length scale. The bending of light is anisotropic (polar angle dependent), as is the focussing effect of the disc (Cai and Shu 2002). Therefore, more realistic geometric distributions of the baryonic matter inside the galaxy and of the exterior space-time geometry of the brane can easily explain the flattening effect and the anisotropic lensing observed for dark matter halos.

One of the main advantages of the brane world models, as compared to the other alternative explanations of the galactic rotation curves and of the dark matter, is that it can give a

systematic and coherent description of the Universe from galactic to cosmological scales. On the other hand, in the present model all the relevant physical quantities, including the dark energy and the dark pressure, which describe the non-local effects due to the gravitational field of the bulk, are expressed in terms of observable parameters (the baryonic mass, the radius of the galaxy and the observed flat rotational velocity curves). Therefore this opens the possibility of testing the brane world models by using astronomical and astrophysical observations at the galactic scale. In this paper we have provided some basic theoretical tools necessary for the in depth comparison of the predictions of the brane world model and of the observational results.

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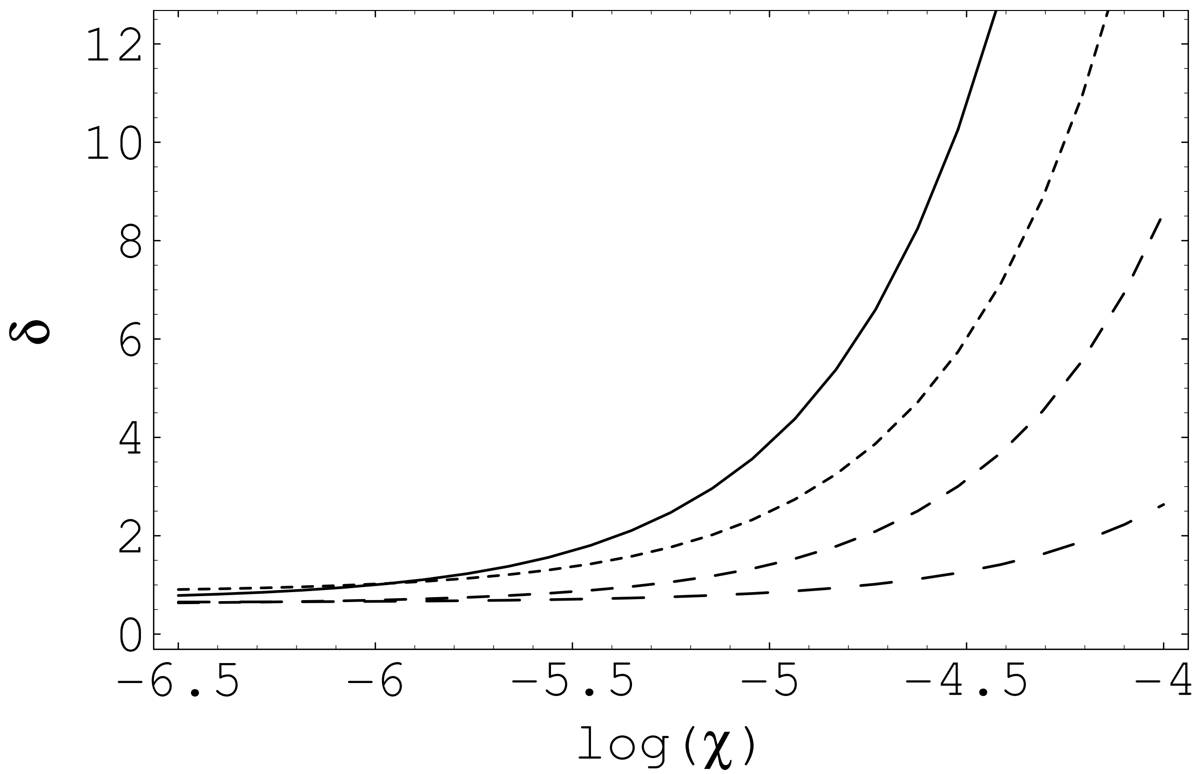

Fig. 1.— The ratio $\delta = (\Delta\phi)_{brane}/2\pi v_{tg}^2$ of the light deflection angle in the galactic metric on the brane and the deflection angle in the semi-realistic isothermal galactic dark matter model as a function of the parameter $\chi = 4GM_B/r_0$ for different values of the tangential velocity: $v_{tg} = 150$ km/s (solid curve), $v_{tg} = 210$ km/s (dotted curve), $v_{tg} = 300$ km/s (short dashed curve) and $v_{tg} = 450$ km/s (long dashed curve). For the impact parameter r_0 and the cosmological constant Λ we have adopted the values $r_0 = 30$ kpc and $\Lambda = 3 \times 10^{-56}$ cm^{-2}, respectively.

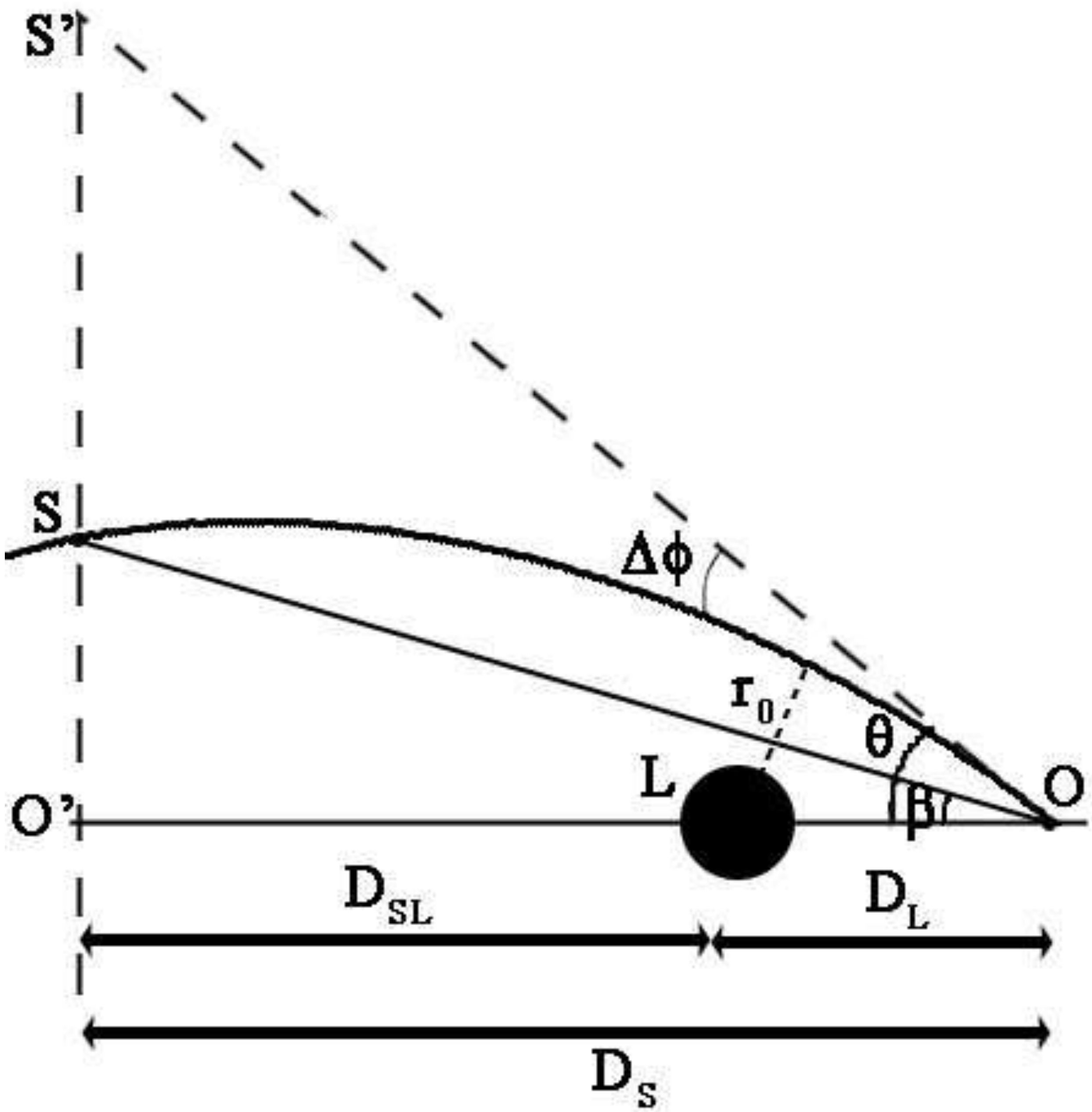

Fig. 2.— The lensing geometry, showing the location of the observer O, the lensing galaxy L and the source S. The deflection angle is $\Delta\phi$. The angular diameter distances D_L, D_{LS} and D_S are also indicated.

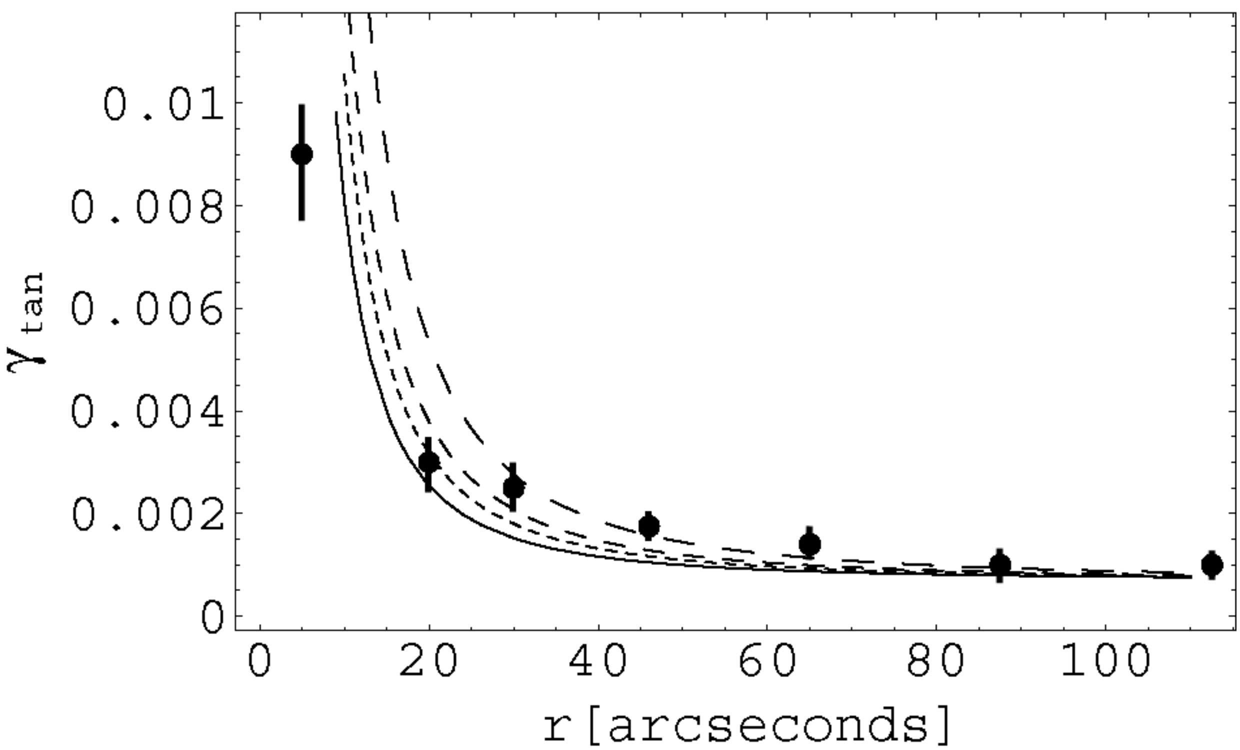

Fig. 3.— Comparison of the variation of the tangential shear as a function of the distance in the brane world model and the observational data of Hoekstra et al. (2004) (represented by points) for different values of the parameter δ: $\delta = 6$ (solid curve), $\delta = 8$ (dotted curve), $\delta = 10$ (dashed curve), $\delta = 12$ (long dashed curve). The Einstein radius is $\theta_E = 0.''14$.